On the Symmetries of Classical String Theory

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Abstract

I discuss some aspects of conformal defects and conformal interfaces in two spacetime dimensions. Special emphasis is placed on their role as spectrum-generating symmetries of classical string theory. Contributed to the volume celebrating Claudio Bunster's sixtieth anniversary; based on talks at the Arnold Sommerfeld Workshop on "String Field Theory and Related Aspects", and for the 50th anniversary of the IHES .

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1 Loop operators in 2d CFT

Wilson loops [47] are important tools for the study of gauge theory. They describe worldlines of external probes, such as the heavy quarks of QCD, which transform in some representation of the gauge group and couple to the gauge fields minimally. More general couplings, possibly involving other fields (e.g. scalars and fermions), are in principle also allowed. They are, however, severely limited by the requirement of infrared relevance or, equivalently, of renormalizability. In four dimensions this only allows couplings to operators of dimension at most one, i.e. linear in the gauge and the scalar fields. An example in which the scalar coupling plays a role is the supersymmetric Wilson loop of $N = 4$ super-Yang Mills theory [38, 43].

The story is much richer in two space-time dimensions. Power-counting renormalizable defects in a two-dimensional non-linear sigma model, for example, are described by the following loop operators

$$\mathrm{tr}_V W(C) = \mathrm{tr}_V \, \mathrm{P}\, e^{i \oint_C \mathrm{H}_{\mathrm{def}}} \quad , \tag{1}$$

where V is the n-dimensional space of quantum states of the external probe, whose Hamiltonian is of the general form

$$\oint_C \mathrm{H}_{\mathrm{def}} = \int ds \left[\left(\vec{\mathbf{B}}_M(\Phi) \partial_\alpha \Phi^M + \varepsilon_{\alpha\beta} \widetilde{\mathbf{B}}_M(\Phi) \partial^\beta \Phi^M \right) \frac{d\hat{\zeta}^\alpha}{ds} + \mathbf{T}(\Phi) \right] . \tag{2}$$

Here s is the length along the defect worldline C, and the Hamiltonian is a hermitean $n \times n$ matrix which depends on the sigma-model fields $\Phi^M(\zeta^\alpha)$, and on their first derivatives, evaluated at the position of the defect $\hat{\zeta}^\alpha(s)$. The loop operator is thus specified by two matrix-valued one-forms, $\mathbf{B}_M d\Phi^M$ and $\widetilde{\mathbf{B}}_M d\Phi^M$, and by a matrix-valued function, $\mathbf{T}$, all defined on the sigma-model target space $\mathscr{M}$. Because $\mathrm{H}_{\mathrm{def}}$ is a matrix, the path-ordering in (1) is non-trivial even if the bulk fields are treated as classical, and hence as commuting c-numbers.

The non-linear sigma model is classically scale-invariant. The function $\mathbf{T}$, on the other hand, has naive scaling dimension of mass, so (classical) scale-invariance requires that we set it to zero. The reader can easily check that, in this case, the operator (1) is invariant under all conformal transformations that preserve C. This symmetry is further enhanced if, as a result of the field equations, the induced one-form

$$\widehat{B} \equiv \left(\mathbf{B}_M(\Phi) \partial_\alpha \Phi^M + \varepsilon_{\alpha\beta} \widetilde{\mathbf{B}}_M(\Phi) \partial^\beta \Phi^M \right) d\zeta^\alpha \tag{3}$$

is a flat $U(n)$ connection, i.e. if in short-hand notation $d\widehat{B} + [\widehat{B}, \widehat{B}] = 0$. The loop operator is in this case invariant under arbitrary continuous deformations of C, as follows from the non-abelian Stoke's theorem. Such defects can therefore be called *topological*. The eigenvalues of topological loops $W(C)$, with C winding around compact space, are charges conserved by the time evolution. The existence of a one-(spectral-)

parameter family of flat connections is, for this reason, often tantamount to classical integrability, see e.g. [6].

Quantization breaks, in general, the scale invariance of the defect loop even when the bulk theory is conformal. This is because the definition of $W(C)$ requires the introduction of a short-distance cutoff ε. As the cutoff is being removed the couplings run to infrared fixed points, $\mathbf{B}^{(\varepsilon)} \to \mathbf{B}^*$ and $\widetilde{\mathbf{B}}^{(\varepsilon)} \to \widetilde{\mathbf{B}}^*$ as $\varepsilon \to 0$. I will explain later that this renormalization-group flow can be described perturbatively [9] by generalized Dirac-Born-Infeld equations. The fixed-point operators commute with a diagonal conformal algebra. More specifically, if C is the circle of a cylindrical spacetime, and L_N and $\overline{L}_N$ are the left- and right-moving Virasoro generators, then

$$[L_N - \overline{L}_{-N}\,,\, \mathrm{tr}_V W^*(C)\,] = 0 \quad \forall N\ . \tag{4}$$

There exists another class of loop operators that commute with the $\overline{L}_N$ (but not necessarily with the L_N) and which we will call *chiral*. Topological operators lie at the intersection of the above two classes: they commute separately with the L_N and the $\overline{L}_N$, and they are thus both conformal and chiral.

All this can be illustrated with the symmetry-preserving defect loops of the WZW model [9]. Consider the following *chiral*, symmetry-preserving defect:

$$\mathscr{O}_r(C) = \chi_r(\,\mathrm{P}\,e^{i\oint_C \lambda J^a t^a}\ )\ , \tag{5}$$

where J^a are the left-moving Kac-Moody currents, t^a the generators of the global group G, and χ_r the character of the G-representation, r, carried by the state-space of the defect. In the classical theory $\mathscr{O}_r(C)$ is topological for all values of the parameter λ. But upon quantization, the spectral parameter runs from the UV fixd point $\lambda^* = 0$ to an IR fixed point $\lambda^* \simeq 1/k$, where k is the level of the Kac-Moody algebra (and $k \gg 1$ for perturbation theory to be valid). It is interesting here to note [9] that one can regularize (5) while preserving the following symmetries: (a) chirality, i.e. $[\mathscr{O}^\varepsilon_r(C), \overline{J}^a_N] = 0$ for all right-moving Kac-Moody (and Virasoro) generators, (b) translations on the cylinder, i.e. $[\mathscr{O}^\varepsilon_r(C), L_0 \pm \overline{L}_0] = 0$, and (c) global $G_{\rm left}$-invariance. These imply, among other things, that the RG flow can be restricted to the single parameter λ, and that the IR fixed-point loop operator is topological. This fixed-point operator is the quantum-monodromy matrix of the WZW model [4]. It can be constructed explicitly, to all orders in the $1/k$ expansion, as a central element of the enveloping algebra of the Kac-Moody algebra [3, 32].

The above renormalization-group flow describes, for $G = SU(2)$, the screening of a magnetic impurity interacting with the left-moving spin current in a quantum wire. This is the celebrated Kondo problem [48][2] which can be solved exactly by the Bethe ansatz [5, 46]. It was first rephrased in the language of conformal field theory by Affleck [1]. Close to the spirit of our discussion here is also the work of Bazhanov et

[2]Strictly-speaking, in the Kondo setup the magnetic impurity interacts with the s-wave conduction electrons of a 3D metal. This is mathematically identical to the problem discussed here.

al [11–13], who proposed to study quantum loop operators in minimal models using conformal (as opposed to integrable lattice-model) techniques. Topological loop operators were first introduced and analyzed in CFT by Petkova and Zuber [40]. Working directly in the CFT makes it possible to use the powerful (geometric and algebraic) tools that were developed for the study of D-branes.

2 Interfaces as spectrum-generating symmetries

Conformal defects in a sigma model with target $\mathscr{M}$ can be mapped to conformal boundaries in a model with target $\mathscr{M} \otimes \mathscr{M}$ by the folding trick [8, 39], i.e. by folding space so that all bulk fields live on the same side of the defect. Conformal boundaries can, in turn, be described either as geometric D-branes [41], or algebraically as conformal boundary states on the cylinder [17, 42]. In the latter description space is taken to be a compact circle, and the boundary state is a (generally entangled) state of the two decoupled copies of the conformal theory:

$$\| \mathscr{B} \rangle\rangle \;=\; \sum \mathscr{B}_{\alpha_1 \tilde{\alpha}_1 a_2 \tilde{\alpha}_2} \, |\alpha_1, \tilde{\alpha}_1\rangle \otimes |\alpha_2, \tilde{\alpha}_2\rangle \; . \tag{6}$$

Here α_j ($\tilde{\alpha}_j$) labels the state of the left- (right-) movers in the jth copy. Unfolding reverses the sign of time for one copy, and thus transforms the corresponding states by hermitean conjugation. This converts $\| \mathscr{B} \rangle\rangle$ to a formal operator, $\mathscr{O}$, which acts on the Hilbert space $\mathscr{H}$ of the conformal field theory. The fixed-point operators of the previous section are all, in principle, unfolded boundary states.

This discussion can be extended readily to the case where the theories on the left and on the right of the defect are different, CFT1 $\neq$ CFT2. Such defects should be, more properly, called *interfaces* or domain walls. They can be described similarly by a boundary state of CFT1$\otimes$ CFT2, or by the corresponding unfolded operator $\mathscr{O}_{21} : \mathscr{H}_1 \to \mathscr{H}_2$. Conformal interfaces correspond to operators that intertwine the action of the diagonal Virasoro algebra,

$$(L_N^{(2)} - \overline{L}_{-N}^{(2)}) \mathscr{O}_{21} \;=\; \mathscr{O}_{21} (L_N^{(1)} - \overline{L}_{-N}^{(1)}) \; , \tag{7}$$

while topological interfaces intertwine separately the action of the left- and right-movers. In the string-theory literature conformal interfaces were first studied as holographic duals [8, 18, 20, 37] to codimension-one anti-de Sitter branes [10, 36]. Note that conformal boundaries are special conformal interfaces for which CFT2 is the trivial theory, i.e. a theory with no massless degrees of freedom. If $\mathscr{O}_{1\emptyset}$ is the corresponding operator (where the empty symbol denotes the trivial theory) then conformal invariance implies that $(L_N^{(1)} - \overline{L}_{-N}^{(1)}) \mathscr{O}_{1\emptyset} = 0$.

I now come to the main point of this talk. Consider a closed-string background described by the worldsheet theory CFT1, and let $\mathscr{O}_{1\emptyset}$ correspond to a D-brane in this background. Take the worldsheet to be the unit disk, or equivalently the semi-infinite cylinder, with the boundary described by the above D-brane. Consider also

a conformal interface $\mathscr{O}_{21}$, where CFT2 describes another admissible closed-string background. Now insert this interface at infinity and push it to the boundary of the cylinder, as in figure 1. The operation is, in general, singular except when $\mathscr{O}_{21}$ is a topological interface in which case it can be displaced freely.

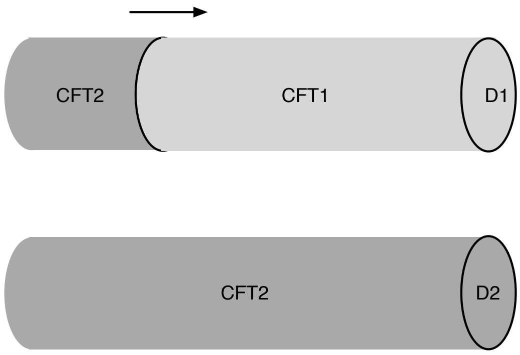

Figure 1: An interface brought from infinity to the boundary of a a cylindrical worldsheet maps the D-branes of one bulk CFT to those of the other. Conformal interfaces between two theories with the same central charge act thus as spectrum-generating symmetries of classical string theory. In many worked-out examples these include and extend the perturbative dualities, and other classical symmetries, of the open- and closed-string action.

Let us assume, more generally, that this fusion operation can be somehow defined and yields a boundary state of CFT2 which we denote by $\mathscr{O}_{21} \circ \mathscr{O}_{1\emptyset}$. We assume that the Virasoro generators commute past the fusion symbol. It follows then from eq. (7) that the new boundary state is conformal whenever the old one was. Since conformal invariance is equivalent to the classical string equations, one concludes that $\mathscr{O}_{21}$ acts as a spectrum-generating symmetry of classical string theory. Conformal interfaces could, in other words, play a similar role as the Ehlers-Geroch transformations [19,27] of General Relativity. Bringing an interface to the boundary is a special case of the more general process of *fusion*, i.e. of juxtaposing and then bringing two interfaces together on the string worldsheet. This is of course only possible when the CFT on the right side of the first interface coincides with the CFT on the left side of the second. Furthermore, two interfaces can only be added when their left and right CFTs are identical. Since fusion and addition cannot be defined for arbitrary elements, the set of all conformal interfaces is neither an algebra nor a group. By abuse of language, I will nevertheless refer to it as the "interface algebra".[3]

The first thing to note is that the interface "algebra" is non-trivial even if restricted only to elements with non-singular fusion. These include all the topological interfaces, for which fusion is the regular product of the corresponding operators,

[3]The correct term for the interfaces is "functors". For a more accurate mathematical terminology the reader should consult, for instance, reference [26].

$\mathscr{O}_A \circ \mathscr{O}_B = \mathscr{O}_A \mathscr{O}_B$. The simplest topological defects are those whose internal state is decoupled from the dynamics in the bulk. They correspond to multiples of the identity operator, $\mathscr{O} = n\mathbf{1}$ with n a natural number. Their action on any D-brane endows this latter with Chan-Paton multiplicity. Less trivial are the topological defects which generate symmetries of the CFT, as well as the topological interfaces that generate perturbative T-dualities. These were first studied, for several examples, in two beautiful papers by Fröhlich et al [23,24]. The fact that all perturbative string symmetries can be realized through the action of local defects is not a priori obvious (and needs still to be generally established). Other interesting examples are the minimal-model topological defects, shown to generate universal boundary flows [22,28]. A different set of conformal interfaces whose fusion is non-singular are those that preserve at least $N = (2,2)$ supersymmetry [14, 15]. Some of these descend from supersymmetric gauge theories in higher dimensions [29, 33–35]. Such interfaces were, in particular, used to generate the monodromy transformations of supersymmetric D-branes transported around singular points in the Calabi-Yau moduli space [16]. As these and other examples demonstrate, the interface "algebra" is very rich even if restricted to interfaces with non-singular fusion.

Extending the structure to arbitrary interfaces is, nevertheless, an interesting problem. Firstly, the algebras (without quotation marks) of non-topological defects would provide, if they could be defined, large extensions of the automorphism groups of various CFTs. Furthermore, while topological interfaces are rare – they may only join CFTs that have isomorphic Virasoro representations – the conformal ones are on the contrary common. To see that conformal interfaces are not rare, consider the nth multiple of the identity defect which is mapped, after folding, to n diagonally-embedded middle-dimensional branes in $\mathscr{M} \times \mathscr{M}$ [8]. A generic Hamiltonian of the form (2), with the tachyon potential $\mathbf{T}$ set to zero, corresponds to arbitrary geometric and gauge-field perturbations of these diagonal branes. Any solution of the (non-abelian, α' corrected) Dirac-Born-Infeld equations for these branes gives therefore rise to a conformal defect [9]. Likewise, any non-factorizable D-brane of CFT1$\otimes$CFT2 unfolds to a non-trivial interface between the two conformal field theories. All of these interfaces can be characterized by a reflection coefficient, $\mathscr{R}$, [44] which must vanish in the topological case.

For most of these interfaces the products of the corresponding operators are singular, so the fusion needs to be appropriately defined. A first step in this direction was taken, in the context of a free-scalar theory, in reference [7]. The rough idea is to define the fusion product as the renormalization-group fixed point to which the system of the two interfaces flows when their separation, ε, goes to zero. A systematic way of doing this, consistent with the distributive property of fusion,[4] has not yet been worked out for interacting theories. For free fields, on the other hand, the story is simpler. The

[4] I thank Maxim Kontsevich for stressing this point.

short-distance singularities are in this case expected to be of the general form

$$\mathscr{O}_A\, e^{-\varepsilon(L_0+\overline{L}_0)}\, \mathscr{O}_B \;\simeq\; \sum_C (e^{2\pi/\varepsilon})^{d_{AB}^{\;C}}\, N_{AB}^{\;C}\, \mathscr{O}_C\;, \tag{8}$$

where $\varepsilon \simeq 0$ is the separation of the two (circular) interfaces on the cylinder, $L_0+\overline{L}_0$ is the translation operator in the middle CFT, the $d_{AB}^{\;C}$ are (non-universal) constants, and the $N_{AB}^{\;C}$ are integer multiplicities. The singular coefficients in the above expression are Boltzmann factors for divergent Casimir energies. The latter must be proportional to $1/\varepsilon$ which is the only scale in the problem (other than the inverse temperature normalized to $\beta = 2\pi$).

By analogy with the operator-product expansion and the Verlinde algebra [45] we may extract from expression (8) the fusion rule

$$\mathscr{O}_A \circ \mathscr{O}_B \;=\; \sum_C N_{AB}^{\;C}\, \mathscr{O}_C\;. \tag{9}$$

The following iterative argument shows that this definition respects the conformal symmetry: first multiply the left-hand-side of (8) with the most singular inverse Boltzmann factor (the one with the largest $d_{AB}^{\;C}$) and take the limit $\varepsilon \to 0$ so as to extract the leading term of the product. Since $[L_N - \overline{L}_{-N}, e^{-\varepsilon(L_0+\overline{L}_0)}] \simeq o(\varepsilon)$ the result commutes with the diagonal Virasoro algebra. Next subtract the leading term from the left-hand-side of (8), and mutliply by the inverse Boltzmann factor with the second-largest $d_{AB}^{\;C}$. This picks up the subleading term which, thanks to the above argument and the conformal symmetry of the leading term, commutes also with the diagonal Virasoro algebra. Continuing this iterative reasoning proves that the right-hand-side of (9) is conformal as claimed.

3 The $c = 1$ CFT and a black hole analogy

A simple context in which to illustrate the above ideas is the $c = 1$ conformal theory of a periodically-identified free scalar field, $\phi = \phi + 2\pi R$. Consider the interfaces that preserve a $U(1)\times U(1)$ symmetry, i.e. those described by linear gluing conditions for the field ϕ. They correspond, after folding, to combinations of D1-branes and of magnetized D2-branes on the orthogonal two-torus whose radii, R_1 and R_2, are the radii on either side of the interface. The D1-branes are characterized by their winding numbers, k_1 and k_2, and by the Wilson line and periodic position moduli α and β. The magnetized D2-branes are obtained from the D1-branes by T-dualizing one of the two directions of the torus – they have therefore the same number of discrete and of continuous moduli.

Let us focus here on the D1-branes. The corresponding boundary states read

$$\|\,\mathrm{D1},\vartheta\rangle\rangle = g^{(+)} \prod_{n=1}^{\infty} (e^{S_{ij}^{(+)} a_n^i \tilde{a}_n^j})^{\dagger} \sum_{N,M=-\infty}^{\infty} e^{iN\alpha - iM\beta} |k_2N, k_1M\rangle \otimes |-k_1N, k_2M\rangle\;, \tag{10}$$

where a_n^j and $\tilde{a}_n^j$ are the left- and right-moving annihilation operators of the field ϕ_j (for $j = 1,2$) and the dagger denotes hermitean conjugation. The ground states $|m,\tilde{m}\rangle$ of the scalar fields are characterized by a momentum (m) and a winding number ($\tilde{m}$). The states in the above tensor product correspond to ϕ_1 and ϕ_2. Furthermore

$$S^{(+)} = \mathscr{U}^T(\vartheta)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathscr{U}(\vartheta) = \begin{pmatrix} -\cos 2\vartheta & -\sin 2\vartheta \\ -\sin 2\vartheta & \cos 2\vartheta \end{pmatrix}, \tag{11}$$

where $\mathscr{U}(\vartheta)$ is a rotation matrix and $\vartheta = \arctan(k_2R_2/k_1R_1)$ is the angle between the D1-brane and the ϕ_1 direction. Finally, the normalization constant is the g-factor [2] of the boundary state. It is given by

$$g^{(+)} = \frac{\ell}{\sqrt{2V}} = \sqrt{\frac{k_1^2R_1^2 + k_2^2R_2^2}{2R_1R_2}} = \sqrt{\frac{k_1k_2}{\sin 2\vartheta}}, \tag{12}$$

where ℓ is the length of the D1-brane, V the volume of the two-torus, and the last rewriting follows from straightforward trigonometry. The logarithm of the g factor is the invariant entropy of the interface.

Inspection of the expression (10) shows that the non-zero modes of the fields ϕ_j are only sensitive to the angle ϑ, which also determines the reflection coefficient of the interface [44]. For fixed k_1 and k_2 the g factor is minimal when $\vartheta = \pm\pi/4$, in which case the reflection $\mathscr{R} = 0$ and the interface is topological. Note that this requirement fixes the ratio of the two bulk moduli: $R_1/R_2 = |k_2/k_1|$. When $|k_1| = |k_2| = 1$ the two radii are equal and the invariant entropy is zero. The corresponding topological defects generate the automorphisms of the CFT, i.e. sign flip of the field ϕ and separate translations of its left- and right-moving pieces. The identity defect corresponds to the diagonal D1-brane, with $k_1 = k_2 = 1$ and $\alpha = \beta = 0$. A T-duality along ϕ_1 maps this topological defect to a D2-brane with one unit of magnetic flux. The corresponding interface operator is the generator of the radius-inverting T-duality transformation. All other topological interfaces have positive entropy, $\log g = \log\sqrt{|k_1k_2|} > 0$. One may conjecture that the following statement is more generally true [7]: the entropy of all topological interfaces is non-negative, and it vanishes only for CFT automorphisms.

The interfaces given by equations (10) to (12) exist for all values of the bulk radii R_1 and R_2. By choosing the radii to be equal we obtain a large set of conformal defects whose algebra is an extension of the automorphism group of the CFT. For a detailed derivation of this algebra see reference [7]. The fusion rule for the discrete defect moduli turns out to be multiplicative,

$$[k_1,k_2;s]\circ[k_1',k_2';s'] \;=\; [k_1k_1',k_2k_2';ss'] ,$$

where $[k_1,k_2;s]$ denotes a defect with integer moduli k_1, k_2, s, where $s = +,-$ according to whether the folded defect is a D1-brane or a magnetized D2-brane. The above fusion rule continues to hold for general interfaces, i.e. when the radii on either side

are not the same. Let me also give the composition rule for the angle ϑ in this general case (assuming $s = s' = +$):

$$\tan(\vartheta \circ \vartheta') = \tan\vartheta \tan\vartheta' \ , \tag{13}$$

where $\vartheta \circ \vartheta'$ denotes the angle of the fusion product. The composition rule (13) was first derived, for the topological interfaces, in reference [25]. In this case the tangents in the last equation are ± 1 and all operator products are non-singular.

There exist some intriguing similarities [7] between the above conformal interfaces and supergravity black holes. The counterpart of BPS black holes are the topological interfaces, which (a) minimize the free energy for fixed values of the discrete charges, (b) fix through an "attractor mechanism" [21] a combination of the bulk moduli, and (c) are marginally stable against dissociation – the inverse process of fusion. The interface "algebra" is, in this sense, reminiscent of an earlier effort by Harvey and Moore [30] to define an extended symmetry algebra for string theory. Their symmetry generators were vertex operators for supersymmetric states of the compactified string. One noteworthy difference is that the additively-conserved charges in the above example are logarithms of natural numbers, rather than taking values in a charge lattice as in [30]. Whether these observations have any deeper meaning remains to be seen. Another direction worth exploring is a possible relation of the above ideas with efforts to formulate string theory in a "doubled geometry", see for instance [31]. The doubling of spacetime after folding suggests that such a formalism may be the natural language in which the defect algebras should be formulated and discussed.

Time now to conclude: conformal interfaces and defects are examples of extended operators, which are a rich and still only partially-explored chapter of quantum field theory. They describe a variety of critical phenomena in low-dimensional condensed-matter systems which, for lack of time, I have not discussed. Conformal interfaces can be added and, at least in many studied examples, juxtaposed or fused. The resulting interface "algebra" defines a large extension of the classical string symmetries, which deserves to be studied more.

Acknowledgements
I thank Ilka Brunner, Jürg Fröhlich and Samuel Monnier for very pleasant collaborations during the last couple of years, on different aspects of this talk. Many thanks also to Eric D'Hoker, Mike Douglas, Sergei Gukov, Chris Hull and Maxim Kontsevich for useful conversations and comments. Claudio Bunster supervised my senior-year undergraduate thesis, and helped me publish my first scientific article – always a source of considerable pride for a student. My gratitude, after all these years, remains intact. This work has been supported in part by the European Community Human Potential Program under contracts MRTN-CT-2004-005104 and MRTN-CT-2004-512194.

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